دانشگاه یزد

هیدرولیک محاسباتی

معادله دوبعدی جابجایی-پخش

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دانشگاه یزد- دانشکده مهندسی عمران

معادله دو بعدی جابجایی-پخش غیرماندگار

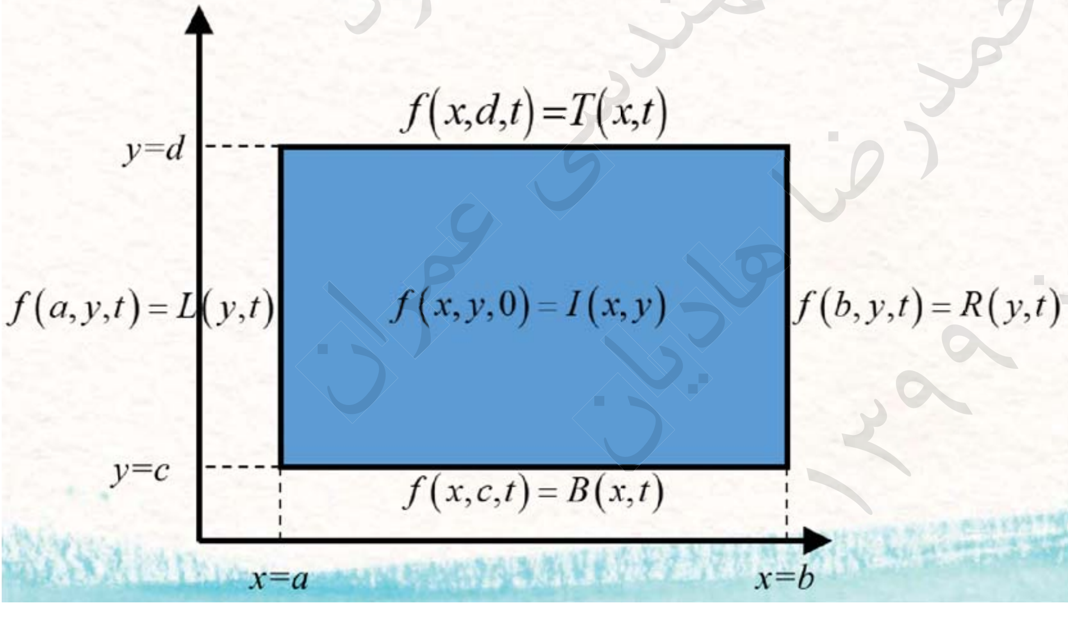
❑ معادله حاکم:

$$f_t + uf_x + vf_y - \alpha_x f_{xx} - \alpha_y f_{yy} = 0$$

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} - \alpha_x \frac{\partial^2 f}{\partial x^2} - \alpha_y \frac{\partial^2 f}{\partial y^2} = 0$$

$$b \ge x \ge a$$

$$d \ge y \ge c$$



$$\Delta x = \frac{b-a}{M-1}$$

$$i = 1 \cdots M$$

$$\Delta y = \frac{d-c}{N-1}$$

$$j = 1 \cdots N$$

Hindmarsh (FTCSCS)

❑ انتخاب فرمول مناسب برای مشتق‌ها

$$f_t + uf_x + vf_y - \alpha_x f_{xx} - \alpha_y f_{yy} = 0$$

$$f_t\Big|_{i,j}^{n} + u\, f_x\Big|_{i,j}^{n} + v\, f_y\Big|_{i,j}^{n} - \alpha_x\, f_{xx}\Big|_{i,j}^{n} - \alpha_y\, f_{yy}\Big|_{i,j}^{n} = 0$$

❖ روش پیش‌رو مرتبه یک برای مشتق زمانی

$$f_t\Big|_{i,j}^{n+1} \cong \frac{f_{i,j}^{n+1} - f_{i,j}^{n}}{\Delta t}$$

❖ روش مرکزی برای مشتق‌های مکانی

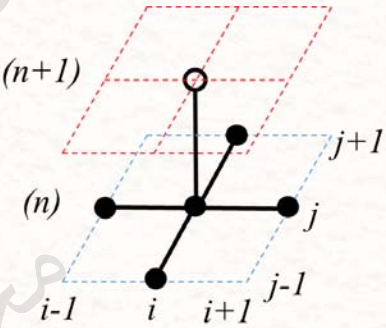
$$f_x\Big|_{i,j}^{n} \cong \frac{f_{i+1,j}^{n} - f_{i-1,j}^{n}}{2\Delta x} \qquad f_{xx}\Big|_{i,j}^{n} \cong \frac{f_{i-1,j}^{n} - 2f_{i,j}^{n} + f_{i+1,j}^{n}}{(\Delta x)^2}$$

$$f_y\Big|_{i,j}^{n} \cong \frac{f_{i,j+1}^{n} - f_{i,j-1}^{n}}{2\Delta y} \qquad f_{yy}\Big|_{i,j}^{n} \cong \frac{f_{i,j-1}^{n} - 2f_{i,j}^{n} + f_{i,j+1}^{n}}{(\Delta y)^2}$$

Hindmarsh (FTCSCS)

$$\frac{f_{i,j}^{n+1} - f_{i,j}^{n}}{\Delta t} + u\frac{f_{i+1,j}^{n} - f_{i-1,j}^{n}}{2\Delta x} + v\frac{f_{i,j+1}^{n} - f_{i,j-1}^{n}}{2\Delta y}$$

$$-\alpha_x \frac{f_{i-1,j}^{n} - 2f_{i,j}^{n} + f_{i+1,j}^{n}}{(\Delta x)^2} - \alpha_y \frac{f_{i,j-1}^{n} - 2f_{i,j}^{n} + f_{i,j+1}^{n}}{(\Delta y)^2} = 0$$



$$S_x = \alpha_x \frac{\Delta t}{(\Delta x)^2} \qquad S_y = \alpha_y \frac{\Delta t}{(\Delta y)^2} \qquad C_x = \frac{u\Delta t}{\Delta x} \qquad C_y = \frac{v\Delta t}{\Delta y}$$

$$f_{i,j}^{n+1} = \left(\frac{C_x}{2} + S_x\right) f_{i-1,j}^{n} + \left(\frac{C_y}{2} + S_y\right) f_{i,j-1}^{n}$$

$$+\left(1 - 2S_x - 2S_y\right) f_{i,j}^{n} - \left(\frac{C_x}{2} - S_x\right) f_{i+1,j}^{n} - \left(\frac{C_y}{2} - S_y\right) f_{i,j+1}^{n}$$

Hindmarsh (FTCSCS)

❑ حوزه پایداری:

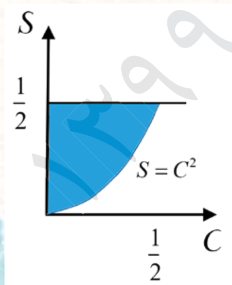
$$\begin{cases} S_x + S_y \le \dfrac{1}{2} \\ \dfrac{C_x^2}{S_x} + \dfrac{C_y^2}{S_y} \le 2 \end{cases}$$

❑ برای حالت خاص:

$$\begin{cases} S_x = S_y = S \\ C_x = C_y = C \end{cases}$$

$$f_{i,j}^{n+1} = \left(\frac{C}{2} + S\right)\left(f_{i-1,j}^{n} + f_{i,j-1}^{n}\right) + (1-4S)\, f_{i,j}^{n} - \left(\frac{C}{2} - S\right)\left(f_{i+1,j}^{n} + f_{i,j+1}^{n}\right)$$

$$\begin{cases} \dfrac{1}{4} \ge S > 0 \\ 1 \ge \dfrac{C^2}{S} \end{cases} \quad \Rightarrow \quad \frac{1}{4} \ge S \ge C^2$$



Upwind explicit (FTBSCS)

❑ انتخاب فرمول مناسب برای مشتق‌ها

$$f_t + u f_x + v f_y - \alpha_x f_{xx} - \alpha_y f_{yy} = 0$$

$$f_t\big|_{i,j}^{n} + u\, f_x\big|_{i,j}^{n} + v\, f_y\big|_{i,j}^{n} - \alpha_x\, f_{xx}\big|_{i,j}^{n} - \alpha_y\, f_{yy}\big|_{i,j}^{n} = 0$$

❖ روش پیش‌رو مرتبه یک برای مشتق زمانی

$$f_t\big|_{i,j}^{n+1} \cong \frac{f_{i,j}^{n+1} - f_{i,j}^{n}}{\Delta t}$$

❖ روش مرکزی برای پخش و بادسو برای جابجایی

$$f_x\big|_{i,j}^{n} \cong \frac{f_{i,j}^{n} - f_{i-1,j}^{n}}{\Delta x} \qquad f_{xx}\big|_{i,j}^{n} \cong \frac{f_{i-1,j}^{n} - 2 f_{i,j}^{n} + f_{i+1,j}^{n}}{(\Delta x)^2}$$

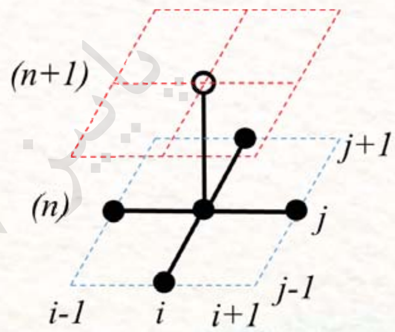
$$f_y\big|_{i,j}^{n} \cong \frac{f_{i,j}^{n} - f_{i,j-1}^{n}}{\Delta y} \qquad f_{yy}\big|_{i,j}^{n} \cong \frac{f_{i,j-1}^{n} - 2 f_{i,j}^{n} + f_{i,j+1}^{n}}{(\Delta y)^2}$$

Upwind explicit (FTBSCS)

$$\frac{f_{i,j}^{n+1}-f_{i,j}^{n}}{\Delta t}+u\frac{f_{i,j}^{n}-f_{i-1,j}^{n}}{\Delta x}+v\frac{f_{i,j}^{n}-f_{i,j-1}^{n}}{\Delta y}$$

$$-\alpha_x\frac{f_{i-1,j}^{n}-2f_{i,j}^{n}+f_{i+1,j}^{n}}{(\Delta x)^2}-\alpha_y\frac{f_{i,j-1}^{n}-2f_{i,j}^{n}+f_{i,j+1}^{n}}{(\Delta y)^2}=0$$

$$f_{i,j}^{n+1}=(C_x+S_x)f_{i-1,j}^{n}+(C_y+S_y)f_{i,j-1}^{n}$$
$$+(1-2S_x-2S_y-C_x-C_y)f_{i,j}^{n}+S_x f_{i+1,j}^{n}+S_y f_{i,j+1}^{n}$$



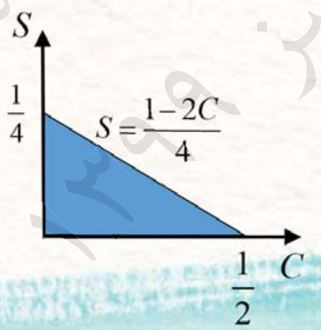
Upwind explicit (FTBSCS)

- حوزه پایداری: $2S_x+2S_y+C_x+C_y\le 1$

- برای حالت خاص: $\begin{cases}S_x=S_y=S\\ C_x=C_y=C\end{cases}$

$$f_{i,j}^{n+1}=(C+S)\left(f_{i-1,j}^{n}+f_{i,j-1}^{n}\right)+(1-4S-2C)f_{i,j}^{n}+S\left(f_{i+1,j}^{n}+f_{i,j+1}^{n}\right)$$

$$\frac{1-2C}{4}\ge S>0$$



Crank-Nicolson

❑ انتخاب فرمول مناسب برای مشتق‌ها

$$f_t + uf_x + vf_y - \alpha_x f_{xx} - \alpha_y f_{yy} = 0$$

$$f_t\Big|_{i,j}^{n+\frac{1}{2}} + u\, f_x\Big|_{i,j}^{n+\frac{1}{2}} + v\, f_y\Big|_{i,j}^{n+\frac{1}{2}} - \alpha_x\, f_{xx}\Big|_{i,j}^{n+\frac{1}{2}} - \alpha_y\, f_{yy}\Big|_{i,j}^{n+\frac{1}{2}} = 0$$

❖ روش مرکزی برای مشتق زمانی

$$f_t\Big|_{i,j}^{n+1} \cong \frac{f_{i,j}^{n+1} - f_{i,j}^{n}}{\Delta t}$$

❖ روش مرکزی برای مشتق‌های مکانی

$$f_x\Big|_{i,j}^{n+\frac{1}{2}} \cong \frac{1}{2}\left(\frac{f_{i+1,j}^{n} - f_{i-1,j}^{n}}{2\Delta x} + \frac{f_{i+1,j}^{n+1} - f_{i-1,j}^{n+1}}{2\Delta x} \right)$$

$$f_y\Big|_{i,j}^{n+\frac{1}{2}} \cong \frac{1}{2}\left(\frac{f_{i,j+1}^{n} - f_{i,j-1}^{n}}{2\Delta y} + \frac{f_{i,j+1}^{n+1} - f_{i,j-1}^{n+1}}{2\Delta y} \right)$$

Crank-Nicolson

$$f_{xx}\Big|_{i,j}^{n+\frac{1}{2}} \cong \frac{1}{2}\left(\frac{f_{i-1,j}^{n} - 2f_{i,j}^{n} + f_{i+1,j}^{n}}{(\Delta x)^2} + \frac{f_{i-1,j}^{n+1} - 2f_{i,j}^{n+1} + f_{i+1,j}^{n+1}}{(\Delta x)^2} \right)$$

$$f_{yy}\Big|_{i,j}^{n+\frac{1}{2}} \cong \frac{1}{2}\left(\frac{f_{i,j-1}^{n} - 2f_{i,j}^{n} + f_{i,j+1}^{n}}{(\Delta y)^2} + \frac{f_{i,j-1}^{n+1} - 2f_{i,j}^{n+1} + f_{i,j+1}^{n+1}}{(\Delta y)^2} \right)$$

$$\frac{f_{i,j}^{n+1} - f_{i,j}^{n}}{\Delta t} + \frac{u}{2}\left(\frac{f_{i+1,j}^{n} - f_{i-1,j}^{n}}{2\Delta x} + \frac{f_{i+1,j}^{n+1} - f_{i-1,j}^{n+1}}{2\Delta x} \right)$$

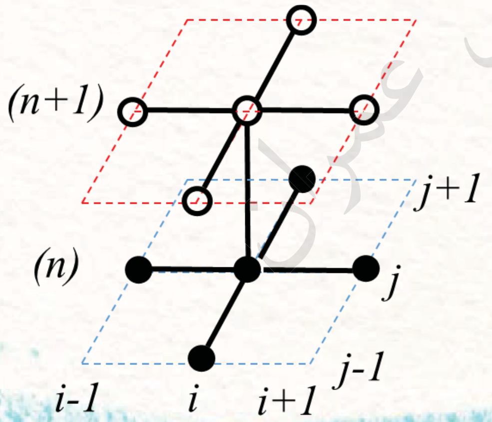
$$+\frac{v}{2}\left(\frac{f_{i,j+1}^{n} - f_{i,j-1}^{n}}{2\Delta y} + \frac{f_{i,j+1}^{n+1} - f_{i,j-1}^{n+1}}{2\Delta y} \right)$$

$$-\frac{\alpha_x}{2}\left(\frac{f_{i-1,j}^{n} - 2f_{i,j}^{n} + f_{i+1,j}^{n}}{(\Delta x)^2} + \frac{f_{i-1,j}^{n+1} - 2f_{i,j}^{n+1} + f_{i+1,j}^{n+1}}{(\Delta x)^2} \right)$$

$$-\frac{\alpha_y}{2}\left(\frac{f_{i,j-1}^{n} - 2f_{i,j}^{n} + f_{i,j+1}^{n}}{(\Delta y)^2} + \frac{f_{i,j-1}^{n+1} - 2f_{i,j}^{n+1} + f_{i,j+1}^{n+1}}{(\Delta y)^2} \right) = 0$$

Crank-Nicolson

$$
\begin{aligned}
&(2S_x+C_x)\, f_{i-1,j}^{n+1} + (2S_y+C_y)\, f_{i,j-1}^{n+1} - 4(1+S_x+S_y)\, f_{i,j}^{n+1} \\
&+(2S_x-C_x)\, f_{i+1,j}^{n+1} + (2S_y-C_y)\, f_{i,j+1}^{n+1} = \\
&-(2S_x+C_x)\, f_{i-1,j}^{n} - (2S_y+C_y)\, f_{i,j-1}^{n} - 4(1-S_x-S_y)\, f_{i,j}^{n} \\
&-(2S_x-C_x)\, f_{i+1,j}^{n} - (2S_y-C_y)\, f_{i,j+1}^{n}
\end{aligned}
$$



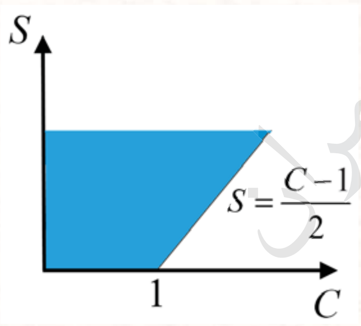
- روش پایدار نامشروط است.

Crank-Nicolson

- برای حالت خاص: $\begin{cases} S_x = S_y = S \\ C_x = C_y = C \end{cases}$

$$
\begin{aligned}
&(2S+C)\left(f_{i-1,j}^{n+1} + f_{i,j-1}^{n+1}\right) - 4(1+2S)\, f_{i,j}^{n+1} + (2S-C)\left(f_{i+1,j}^{n+1} + f_{i,j+1}^{n+1}\right) = \\
&-(2S+C)\left(f_{i-1,j}^{n} + f_{i,j-1}^{n}\right) - 4(1-2S)\, f_{i,j}^{n} - (2S-C)\left(f_{i+1,j}^{n} + f_{i,j+1}^{n}\right)
\end{aligned}
$$

- حوزه حل‌پذیری: $S \geq \dfrac{C-1}{2} > 0$



$$E = O\left\{(\Delta x)^2, (\Delta y)^2, (\Delta t)^2\right\}$$

Time Splitting Method- ADI

$$\left[n\Delta t,(n+1)\Delta t\right]=\left[n\Delta t,(n+\tfrac{1}{2})\Delta t\right]\cup\left[(n+\tfrac{1}{2})\Delta t,(n+1)\Delta t\right]$$

$$\left[n\Delta t,(n+\tfrac{1}{2})\Delta t\right]:\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}-\alpha_x\frac{\partial^2 f}{\partial x^2}-\alpha_y\frac{\partial^2 f}{\partial y^2}=0$$

$$\left[(n+\tfrac{1}{2})\Delta t,(n+1)\Delta t\right]:\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}-\alpha_x\frac{\partial^2 f}{\partial x^2}-\alpha_y\frac{\partial^2 f}{\partial y^2}=0$$

Time Splitting Method- LOD

❑ گزینه اول- تقسیم به دو نیم گام

$$\left[n\Delta t,(n+1)\Delta t\right]=\left[n\Delta t,(n+\tfrac{1}{2})\Delta t\right]\cup\left[(n+\tfrac{1}{2})\Delta t,(n+1)\Delta t\right]$$

$$\left[n\Delta t,(n+\tfrac{1}{2})\Delta t\right]:\quad \frac{1}{2}\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}-\alpha_x\frac{\partial^2 f}{\partial x^2}=0$$

$$\left[(n+\tfrac{1}{2})\Delta t,(n+1)\Delta t\right]:\quad \frac{1}{2}\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial y}-\alpha_y\frac{\partial^2 f}{\partial y^2}=0$$

❖ Lax-Wendroff

$$f_i^{n+\frac{1}{2}}=\frac{2S_x+C_x+C_x^2}{2}f_{i-1,j}^n+\left(1-2S_x-C_x^2\right)f_{i,j}^n+\frac{2S_x-C_x+C_x^2}{2}f_{i+1,j}^n$$

$$f_i^{n+1}=\frac{2S_y+C_y+C_y^2}{2}f_{i,j-1}^{n+\frac{1}{2}}+\left(1-2S_y-C_y^2\right)f_{i,j}^{n+\frac{1}{2}}+\frac{2S_y-C_y+C_y^2}{2}f_{i,j+1}^{n+\frac{1}{2}}$$

Time Splitting Method- LOD

❑ گزینه دوم- تقسیم به چهار گام جزئی

$$\left[n\Delta t,(n+1)\Delta t\right]=\left[n\Delta t,(n+\tfrac{1}{4})\Delta t\right]\cup\left[(n+\tfrac{1}{4})\Delta t,(n+\tfrac{1}{2})\Delta t\right]$$
$$\cup\left[(n+\tfrac{1}{2})\Delta t,(n+\tfrac{3}{4})\Delta t\right]\cup\left[(n+\tfrac{3}{4})\Delta t,(n+1)\Delta t\right]$$

$$\left[n\Delta t,(n+\tfrac{1}{4})\Delta t\right]: \qquad \frac{1}{4}\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}=0$$

$$\left[(n+\tfrac{1}{4})\Delta t,(n+\tfrac{1}{2})\Delta t\right]: \qquad \frac{1}{4}\frac{\partial f}{\partial t}-\alpha_x\frac{\partial^2 f}{\partial x^2}=0$$

$$\left[(n+\tfrac{1}{2})\Delta t,(n+\tfrac{3}{4})\Delta t\right]: \qquad \frac{1}{4}\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial y}=0$$

$$\left[(n+\tfrac{3}{4})\Delta t,(n+1)\Delta t\right]: \qquad \frac{1}{4}\frac{\partial f}{\partial t}-\alpha_y\frac{\partial^2 f}{\partial y^2}=0$$

Time Splitting Method- LOD

❖ برای مراحل ۱ و ۳ از Lax-Wendroff

❖ برای مراحل ۲ و ۴ از FTCS

$$\left[n\Delta t,(n+\tfrac{1}{4})\Delta t\right]: \qquad f_i^{n+\frac{1}{4}}=\frac{C_x+C_x^2}{2}f_{i-1,j}^{n}+\left(1-C_x^2\right)f_{i,j}^{n}+\frac{C_x^2-C_x}{2}f_{i+1,j}^{n}$$

$$\left[(n+\tfrac{1}{4})\Delta t,(n+\tfrac{1}{2})\Delta t\right]: \qquad f_i^{n+\frac{1}{2}}=S_x f_{i-1,j}^{n+\frac{1}{4}}+\left(1-2S_x\right)f_{i,j}^{n+\frac{1}{4}}+S_x f_{i+1,j}^{n+\frac{1}{4}}$$

$$\left[(n+\tfrac{1}{2})\Delta t,(n+\tfrac{3}{4})\Delta t\right]: \qquad f_i^{n+\frac{3}{4}}=\frac{C_y+C_y^2}{2}f_{i,j-1}^{n+\frac{1}{2}}+\left(1-C_y^2\right)f_{i,j}^{n+\frac{1}{2}}+\frac{C_y^2-C_y}{2}f_{i,j+1}^{n+\frac{1}{2}}$$

$$\left[(n+\tfrac{3}{4})\Delta t,(n+1)\Delta t\right]: \qquad f_i^{n+1}=S_y f_{i,j-1}^{n+\frac{3}{4}}+\left(1-2S_y\right)f_{i,j}^{n+\frac{3}{4}}+S_y f_{i,j+1}^{n+\frac{3}{4}}$$

Time Splitting Method- LOD

❑ گزینه سوم- تقسیم به دو نیم گام

$$\left[n\Delta t,(n+1)\Delta t\right]=\left[n\Delta t,(n+\tfrac{1}{2})\Delta t\right]\cup\left[(n+\tfrac{1}{2})\Delta t,(n+1)\Delta t\right]$$

$$\left[n\Delta t,(n+\tfrac{1}{2})\Delta t\right]:\qquad \frac{1}{2}\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}=0$$

$$\left[(n+\tfrac{1}{2})\Delta t,(n+1)\Delta t\right]:\quad \frac{1}{2}\frac{\partial f}{\partial t}-\alpha_x\frac{\partial^2 f}{\partial x^2}-\alpha_y\frac{\partial^2 f}{\partial y^2}=0$$

❖ برای نیم گام اول و دوم با توجه به ماهیت متفاوت ترم‌ها روش‌های متفاوتی استفاده می‌شود.

Time Splitting Method- LOD

❑ آیا ترتیب چیدن معادلات مهم است؟

$$\left[n\Delta t,(n+1)\Delta t\right]=\left[n\Delta t,(n+\tfrac{1}{4})\Delta t\right]\cup\left[(n+\tfrac{1}{4})\Delta t,(n+\tfrac{1}{2})\Delta t\right]$$
$$\cup\left[(n+\tfrac{1}{2})\Delta t,(n+\tfrac{3}{4})\Delta t\right]\cup\left[(n+\tfrac{3}{4})\Delta t,(n+1)\Delta t\right]$$

$$\left[n\Delta t,(n+\tfrac{1}{4})\Delta t\right]:\quad \frac{1}{4}\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}=0$$

$$\left[(n+\tfrac{1}{4})\Delta t,(n+\tfrac{1}{2})\Delta t\right]:\quad \frac{1}{4}\frac{\partial f}{\partial t}-\alpha_x\frac{\partial^2 f}{\partial x^2}=0$$

$$\left[(n+\tfrac{1}{2})\Delta t,(n+\tfrac{3}{4})\Delta t\right]:\quad \frac{1}{4}\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial y}=0$$

$$\left[(n+\tfrac{3}{4})\Delta t,(n+1)\Delta t\right]:\quad \frac{1}{4}\frac{\partial f}{\partial t}-\alpha_y\frac{\partial^2 f}{\partial y^2}=0$$

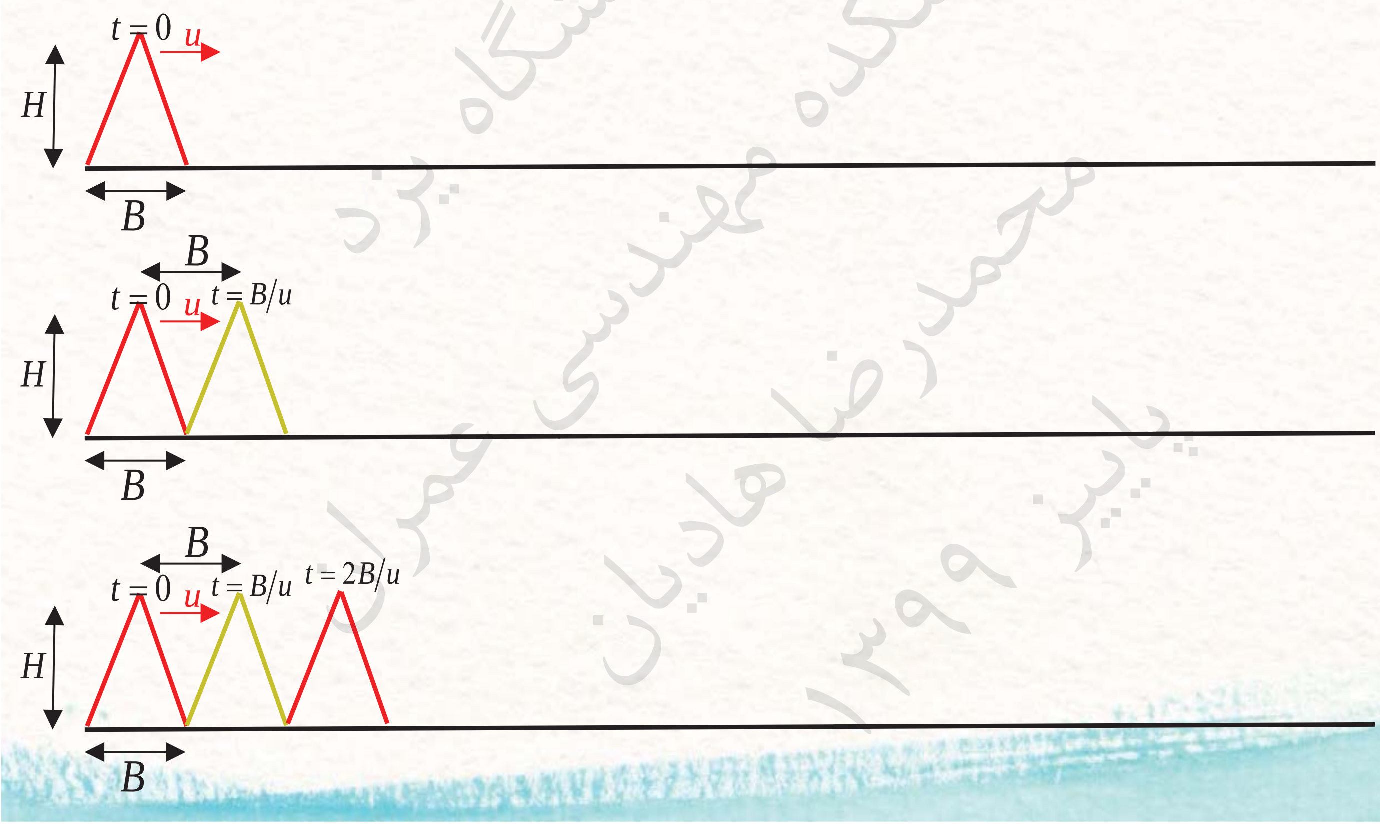
$$\left[n\Delta t,(n+\tfrac{1}{4})\Delta t\right]:\quad \frac{1}{4}\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}=0$$

$$\left[(n+\tfrac{1}{4})\Delta t,(n+\tfrac{1}{2})\Delta t\right]:\quad \frac{1}{4}\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial y}=0$$

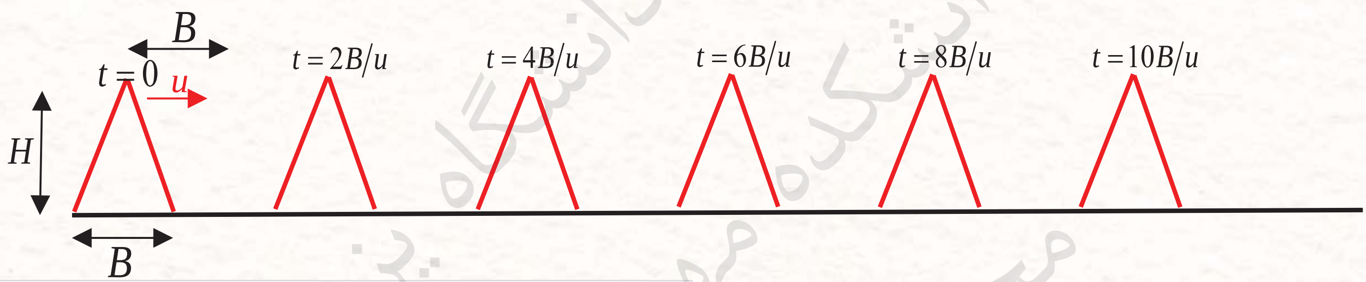
$$\left[(n+\tfrac{1}{2})\Delta t,(n+\tfrac{3}{4})\Delta t\right]:\quad \frac{1}{4}\frac{\partial f}{\partial t}-\alpha_x\frac{\partial^2 f}{\partial x^2}=0$$

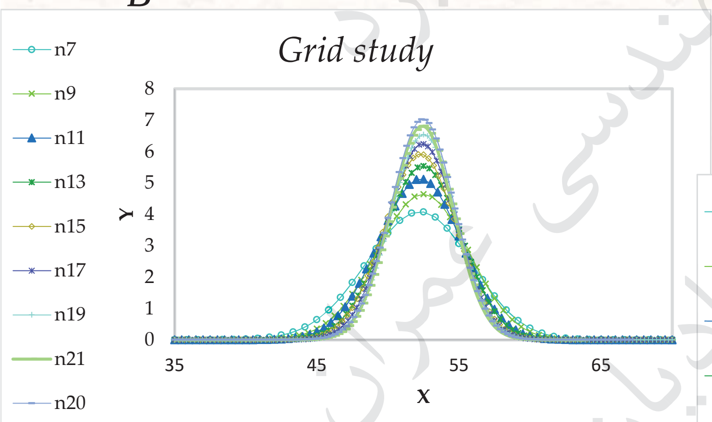
$$\left[(n+\tfrac{3}{4})\Delta t,(n+1)\Delta t\right]:\quad \frac{1}{4}\frac{\partial f}{\partial t}-\alpha_y\frac{\partial^2 f}{\partial y^2}=0$$

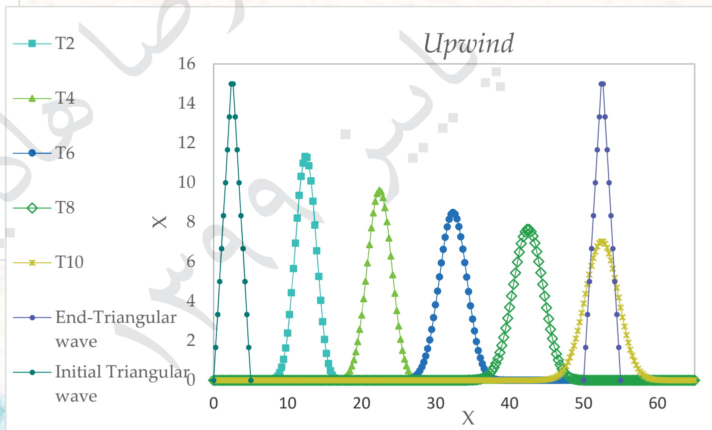
تمرین حل معادله یک‌بعدی جابجایی



تمرین حل معادله یک‌بعدی جابجایی







Any Question?